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Analysis of opinion consensus and fluctuation over networks

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Abstract: The dynamics of opinion formation with two types of individuals over social networks is analyzed in this paper. During each interaction, individuals pick up one of his neighbors with a probability depending on the neighbor's closeness and popularity and update their opinion based on the convex combination of the opinion of the interactive individuals. It is found that when the stubborn individuals have the same initial opinion, the whole population will definitely converge to a consensus, otherwise, opinion fluctuation can be be observed if regular individuals can reach at least two stubborn individuals with different opinion. Moreover, in the small world network larger average degree and rewiring probability can accelerate convergence speed to a large extent especially when they are relative small. The results may have some insight on the opinion dynamics in the real world.

Key Words: Opinion Consensus, Fluctuation, Social Networks, Stubborn Individual

1 Introduction

Research on opinion formation over social network has emerged as a hot interdisciplinary topic in the last decade and various works have been presented to explore this area from the perspective of both mathematical analysis and numerical simulations [1, 2]. For example [3] investigated the dynamics of opinion evolution on crowd-seeking under the influence of disturbance based on mean filed game. The equilibrium and stability analysis are presented to prove that all the opinions can converge to the mean opinion eventually. [4] studied the dynamics opinion evolution based on DW model as well as the influence of network topology and found that convergence time undergoes a phase transition at a critical value of bounded confidence and the final individual opinions change from consensus to multi opinion groups. An empirical analysis of opinion formation on social network is presented [5], which found that the individuals are usually reluctant to change their opinion and the distribution of the number of opinion changes follows a power law. And an agent-based model with external actions is proposed and the probability of the individual to change opinion depends nonlinearly on the fraction of his neighbors to take such action, which can fit the real world data well. [6] proposed an agent-based model where individuals interact with others depending on similarity and popularity and updated their opinon based on weighted average rule, and gave mathematical analysis that the individuals will converge to a consensus in a finite time. [7] proposed social judgement based opinion dynamics model, where interacting individuals will update their opinion based on the rules of both compromise and repulsion. It is found that the extreme opinion may emerge eventually through numerical simulations. Similarly, [8] considered both the positive and negative influence from neighbors and propose a model to investigate the dynamics of opinion polarization. These research mainly focus on how individual opinion eventually converge to a consensus or multi-group opinions.

In the real world, however, opinion fluctuation is often observed. For example Kramer in [9] showed that there was large swing in voting behavior within a short term. Recently Acemoglu et al in [10] analyzed the opinion fluctuation based on gossip model with stubborn agents. Zhang et al in [11] studied the opinion fluctuation based on DW model with long range interaction and indicated that as long as the bounded confidence is larger than a critical value, the regular individual's opinion would keep fluctuating. Wang proposed a second-order opinion model with stubborn agents and regular agents in [12] to investigate opinion consensus and fluctuation and theoretical analysis is also presented.

Up to now, the research on opinion fluctuations is rather rare compared to that about opinion agreement. In this paper, the dynamics of opinion convergence and fluctuation is further investigated through both theoretical analysis and numerical simulations. Inspired by the previous works[10–12], individuals are split into two types: regular individuals and stubborn individuals, and the dynamics of opinion evolution of the population with the same stubborn individual and different stubborn individuals will be analyzed.

The rest of this paper is organized as follows. The model formulation and preliminaries are presented in the following section, while the theoretical analysis is presented in Section 3, and numerical simulations in Section 4. Finally some conclusions are given in the section 5.

2 Model Formulation

Consider a society composed of n individuals, denoted by $V = \{1, 2, \cdots, n\}$. The individuals can be split into two types: stubborn individual, which will be never influenced by others, and regular individual, which will be influenced by others. Denote the set of regular individuals as $\mathcal{A}$ while the set of stubborn individuals as $\mathcal{B}$, i.e. $V = \mathcal{A} \cup \mathcal{B}$. For any individual $i \in V$, denote by $x_i(t) \in [0 \quad 1]$ the value of $i's$ opinion at time $t (t \geq 0)$, and $\mathbf{X}(t) = [x_1(t), x_2(t), \cdots, x_n(t)]^T \in [0 \quad 1]^n$ as the whole

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population's opinion at time t. The individuals interact with each other over a connected social network, represented by $G = (V, E)$, where $E \subset V \times V$ is the set of edges. Assume for all $i \in V$, $(i, i) \notin E$. The neighbor set of individual i is $N_i = \{j \in V : (i, j) \in E\}$.

Individuals starts with an scalar opinion $\mathbf{X}(0) = [x_1(0), x_2(0), \cdots, x_n(0)]^T$. At each time step, each individual $i \in V$ will firstly choose one of his neighbors $j \in N_i(t)$ to exchange opinion information. After the interaction, individual i updates his opinion by weighted averaging his own opinion $x_i(t)$ and the chosen neighbor's opinion $x_j(t)$. Inspired by [6, 13], the probability that j was chosen by i at time t, p_{ij}, depends on how close is between $x_i(t)$ and $x_j(t)$ and how popular individual j is. Thus before presenting the opinion model, the closeness between i and j at time t, $s_{ij}(t)$, and the popularity of j, v_j, should be determined.

The closeness between i and j can be given as Eq.(1).

$$s_{ij}(t) = e^{-|x_i(t-1) - x_j(t-1)|} \tag{1}$$

The popularity can be defined as the frequency of being visited, i.e. if individual j is visited by others more often, his popularity is much higher. Assume the number of visit received by j at time $t-1$ is $d_j(t-1)$, then the popularity of j can be defined as Eq.2:

$$v_j(t) = e^{-\left(1 - \frac{d_j(t-1)}{n-1}\right)} \tag{2}$$

Combining Eq.(1) and Eq.2, the probability with which individual j can be chosen by i at time t, $p_{ij}(t)$, can be determined as (3)

$$p_{ij}(t) = \frac{s_{ij}(t)^{\alpha} v_{ij}(t)^{\beta}}{\sum_{k \in N_i} s_{ik}(t)^{\alpha} v_{ik}(t)^{\beta}} \tag{3}$$

where α, $\beta \geq 0$. α can be thought of the openness of the individual, i.e. larger value of α indicates that individuals are more inclined to interact with those with similar opinion, while smaller value of α indicates that the individuals are more accepting of different opinion. Similarly, individuals with larger β are more likely to interact with popular neighbors than those with smaller β. α and β can be individual-specific, but for simplicity, in this paper they are assumed to be the same for all the individuals. When $\alpha = \beta = 0$, individual i will choose his partner randomly. Since similar analysis of these two parameter has been given in [6], in this paper we will not discuss their impact on the dynamics of opinion evolution. Instead, in the following simulations, we assume that $\alpha = \beta = 2$.

After determining the interactive individual, i will update his opinion $x_i(t+1)$ based on average rule, as is shown in Eq.(4)

$$x_i(t+1) = \begin{cases} x_i(t) \equiv x_i(0), & if \;\; i \in \mathcal{B} \\ x_i(t) + \theta(x_j(t) - x_i(t)), & if \;\; i \in \mathcal{A} \end{cases} \tag{4}$$

where $\theta \in [0, 1]$ is the weight that i places on individual j's opinion.

For all the individuals, the updated opinion can be written in the form of matrix as Eq.(5)

$$\mathbf{X}(t+1) = Z(t)\mathbf{X}(t) \tag{5}$$

where $Z(t) \in \mathbb{R}^{n \times n}$ is the weight matrix for the population to update his opinion. By iteration, the updated opinion of the population can be represented as Eq.(6)

$$\mathbf{X}(t+1) = Z(t)Z(t-1)Z(t-2)\ldots Z(0)\mathbf{X}(0) \tag{6}$$

3 Theoretical Analysis

In what follows, the dynamics of Eq.(6) will be analyzed. Since Eq.6 can be regarded as the combination of the following two cases: *i)* when all of the stubborn individuals have the same initial opinion, i.e. $x_{p+1}(0) = x_{p+2}(0) = \cdots = x_{p+q}(0)$; *ii)* when the stubborn individuals do not have the same initial opinion.

Before proceeding the analysis, some useful observations and results will be presented.

Firstly, as for the property of weight matrix $Z(t)$, recall that in each interaction individual $i \in V$ only interacts with one of his neighbors, denoted by $r_i \in N_i(t)$, the entries of the weight matrix $\{Z : Z_{ij}\}$ can be thus determined as follows:

(i) if $i \in \mathcal{B}$

$$Z_{ij} = \begin{cases} 1, & j = i \\ 0, & j \neq i \end{cases}$$

(ii) if $i \in \mathcal{A}$

$$Z_{ij} = \begin{cases} 1-\theta, & j = i \\ \theta, & j = r_i \\ 0, & j = otherwise \end{cases}$$

Obviously, $Z(t)$ is a stochastic matrix. For convenience, we assume that the number of stubborn individuals as q and that of regular individuals as p, i.e. $p + q = n$. Without loss of generality, let the set of regular individuals as $\mathcal{A} := \{1, 2, \cdots, p\}$ and stubborn individuals as $\mathcal{B} := \{p+1, p+2, \cdots, p+q\}$. Then the diagonal element of $Z(t)$ is $diag(Z) = \{\underbrace{1-\theta, \cdots, 1-\theta}_{p}, \underbrace{1, \cdots, 1}_{q}\}$.

Since $Z(t)$ is state-dependent and not symmetric. The popular methods used to analyze the stability of agent-based system will not be applicable. We will use the result presented in [6] to investigate the dynamics of Eq.(6).

Lemma 1[6]: Let ε be all of the possible outcomes, and $\{E_t\}_{t=0}^{\infty}$ be a sequence of events, not necessarily independent, such that $P(E_t) \geq \gamma$ with $\gamma > 0$ for all $E_t \in \varepsilon$. If $\sum_{t=0}^{\infty} P(E_t) = \infty$, then $P(lim_{t\to\infty} sup E_t) = 1$.

Theorem 1[14]: For any vector $\mathbf{w} = \{w_i\}$ and stochastic matrix $P = \{p_{ij}\}$, if there exists a vector $\mathbf{z} = \{z_i\}$, such that $\mathbf{z} = P\mathbf{w}$, then

$$\left\{\max_j z_j - \min_j z_j\right\} \leq \tau(P)\left\{\max_j w_j - \min_j w_j\right\} \tag{7}$$

Next we will analyze the final state of opinion evolution for case *i)*. Let E_t be the event:

$$E_t : Z_{ij}(t) > 0, (i \in A)$$

Since for any regular individual $i \in A$, the probability with that i visits other individuals $j \in \{V\backslash i\}$ satisfies $p_{ij}(t) \geq \frac{e^{-\beta-\alpha}}{n-1}$, we have $P(Z_{ij}(t) = \theta > 0) \geq \frac{e^{-\beta-\alpha}}{n-1} > 0$, where j is the individual visited by i. In addition the diagonal elements of $Z(t)$ is positive. Then according to the Lemma 1, we can have

$$P(lim_{t\to\infty} supE_t) = 1 \quad (8)$$

Eq.(8) indicates that all of the regular individuals can visit any other individual and different regular individuals can visit the same other individual at one time, i.e. there must exit one column whose elements are positive in the front p rows of matrix $Z(t)$ and such column can be in any position.

Since the stubborn individuals have the same initial opinion, without loss of generality assume $x_i(t) = x_{p+1}(0)$ for all $i \in \mathcal{B}$, i.e. we regard all of the stubborn individuals as the same one. Thus the element of matrix $Z(t)$ can be represented as Eq.(9) for all the stubborn individuals $i \in \mathcal{B}$

$$Z_{ij}(t) = \begin{cases} 1, & j = p+1 \\ 0, & j = \text{otherwise} \end{cases} \quad (9)$$

Eq.(9) indicates that all of the stubborn individuals are mapped into the position of $p+1$ in the matrix of Z. Thus all of the element in the $p+1$th column of matrix Z are positive, $Z(t)$ is a *scrambling matrix*.

According to [14] the coefficient of ergodicity of $Z(t)$ can be defined as follows:

$$\tau(Z(t)) = 1 - \min_{i,j} \sum_{k=1}^{n} min(Z_{ik}(t), Z_{jk}(t)) \quad (10)$$

Let

$$\delta_{Z(t)} = \min_{i,j} \sum_{k=1}^{n} min(Z_{ik}(t), Z_{jk}(t))$$

Since $Z(t)$ is a scrambling matrix, $\delta_{Z(t)} > 0$[14], leading to $\tau(Z(t)) < 1$.

Thus as $t \to \infty$, we have

$$\prod_{k=0}^{t} \tau(Z(k)) = 0, a.s.$$

In addition, by referring to Theorem 1 and Eq.(6), we can have

$$\begin{aligned} &\left\{\max_i x_i(t+1) - \min_i x_i(t+1)\right\} \\ \leq &\tau(Z(t))\left\{\max_i x_i(t) - \min_i x_i(t)\right\} \\ \leq &\prod_{k=0}^{t} \tau(Z(k))\left\{\max_i x_i(0) - \min_i x_i(0)\right\} \end{aligned} \quad (11)$$

Obviously the following equation can be held as $t \to \infty$

$$\left\{\max_i x_i(t+1) - \min_i x_i(t+1)\right\} = 0, a.s. \quad (12)$$

Thus it can be concluded that all of the individuals will make a consensus on $x_{p+1}(0)$.

It should be noted that the above analysis is based on the assumption that the network topology is complete network. However, according to the result of joint connected network, such result is also valid as long as the network is connected. Therefore, we can have the following result:

Theorem 2: Consider a population V interacting over a connected social network, if the stubborn individuals hold the same initial opinion and the population update their opinion based on Eq.(5), then the whole population can reach consensus on the stubborn individuals's initial opinion.

Next we will discuss the case when the stubborn individuals have different opinions, i.e. there exists at least two stubborn individuals $i, j \in \mathcal{B}$, such that $x_i(0) \neq x_j(0)$. Obviously, the element of weight matrix $Z(t)$ cannot be represented as Eq.(9) and $Z(t)$ is not a scrambling matrix. By contradiction, it can be concluded that the population cannot reach a consensus. It is conjectured that the regular individuals will keep fluctuated in the interval where the upper bound is the largest opinion of the reachable stubborn individuals, while the lower bound is the smallest opinion among the reachable stubborn individuals. In the following section, we will validate such conjecture through numerical simulations.

4 Numerical Simulation

In this section the numerical simulations on different network topologies will be presented. For all of the simulation, consider the population size $n = 50$, and assume the parameter $\theta = 0.5$.

4.1 Results of the same stubborn individuals

Assume the number of stubborn individuals $q = 2$ and denote by $\{s_1, s_2\}$ as the stubborn individuals, which is selected randomly from the population and the initial opinion of the stubborn individuals as $x_{s_1} = x_{s_2} = x_s = 0.5$, while that of the regular individuals is uniformly distributed in the interval $[0, 1]$. The time evolution of opinion of the population on complete network is shown in Fig.1, where the bold line is the opinion of stubborn individuals. It can be seen that the opinion of the regular individuals converges to the stubborn individuals eventually. We also conduct the simulation with different x_s and find that the regular individuals can easily converge to x_s too.

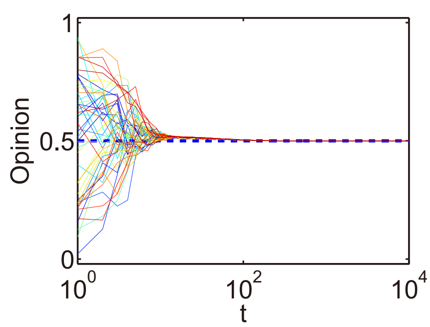

Fig. 1: Time evolution of opinion of the population on the complete network.

Simulations are also carried out on ER random network[15] and small world networks[16] with the same set up. They show the same trend as Fig.1. Here due to the limited space, the results are not presented. Instead, in order to further investigate the influence of network topology on the dynamics of opinion evolution of the population, we will analyzed the impact of network topology on dynamics of opinion evolution, especially with regards to the impact of connection probability of ER random network p and the rewiring probability p_r as well as the average degree of small network network on convergence time of the population, t_c.

As for the connection probability p of random network, assume p varies from 0.1 to 1, the average convergence time t_c with different p is shown in Fig.2, which is an average over 20 independent realizations. It can be observed that the population can reach convergence much faster with larger connection probability, especially when p increases from 0.1 to 0.3. This is mainly due to the fact that the degree of the random network becomes much larger with larger p, leading to shorter average path length. But when p is large enough, its increase has little impact on the convergence speed. That is because if the network is dense enough, the extra increase of network edges cannot increase the probability remarkably for the two individuals to interact with each other. Thus as p continues to increase, the average convergence time keeps almost unchanged.

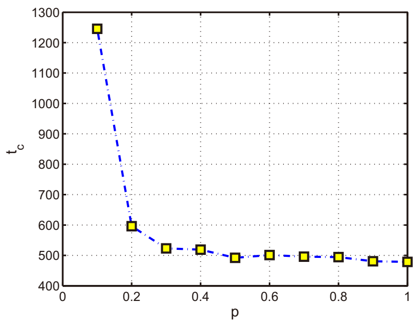

Fig. 2: The average convergence time t_c with different connection probability p of ER random network.

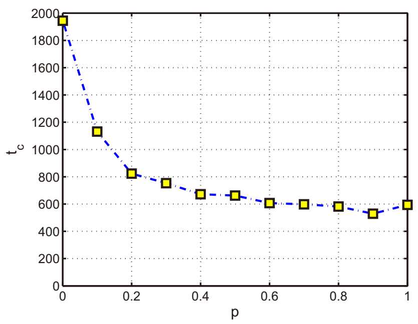

Fig. 3: The average convergence time t_c with different rewiring probability p_r on small world network.

As for the rewiring probability of small world network p_r, assume the average degree of the network $\langle K \rangle = 6$ and rewiring probability p_r increases from 0 to 1, the average convergence speed for the population to reach consensus with different p_r is presented in Fig.3. It can be seen that when p_r is relatively small, the average convergence time t_c reduces dramatically as p_r increases, however, when p_r is large enough (0.4 in this case), the average t_c will remain almost stable as p_r becomes larger. That is because when p_r is large enough, the average path length keeps almost unchanged with larger p_r, as a result, the influence of p_r on the convergence speed is very little.

Similarly, we will look into the influence of the degree of network $\langle K \rangle$ on convergence time. Assume the rewiring probability $p_r = 0.2$ and the average degree $\langle K \rangle$ varies in the interval $[2, 50]$. Fig.4 is the average convergence time t_c with different $\langle K \rangle$, from which it can be observed that t_c decrease quickly and then remains almost stable as $\langle K \rangle$ increases from 2 to 50. This is because when $\langle K \rangle$ becomes larger, the network is much denser and the path length of any two individuals will become less and individuals are much more likely to be chosen as another one's partner compared to the spare social network. Thus the population can converge to the stable state much more quickly.

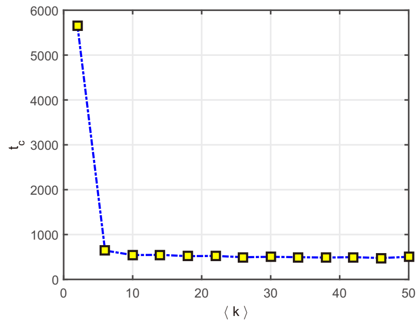

Fig. 4: The average convergence time t_c with different average degree $\langle K \rangle$ of S-W small world network.

4.2 Result of different stubborn individuals

In this subsection we will give the simulation results when the stubborn individuals have different opinion. Consider the number of stubborn individuals $q = 3$, denoted as $\mathcal{B} := \{s_1, s_2, s_3\}$. Assume the initial opinion of the stubborn individual as $s_1 = 0.4$, $s_2 = 0.6$ and $0s_3 = 0.8$, respectively, in the following simulation results.

Fig.5 is opinion evolution of the population on ER random networks, where we assume the stubborn individuals are selected randomly on the condition that all the regular individuals can reach all of the stubborn individual. From Fig.5 it can be seen obviously that regular individuals opinion converge to interval $[0.4, 0.8]$ quickly and then keep fluctuated in that interval, which corresponds to the smallest and biggest value of the stubborn individuals' opinion. The results coincide with the conjecture.

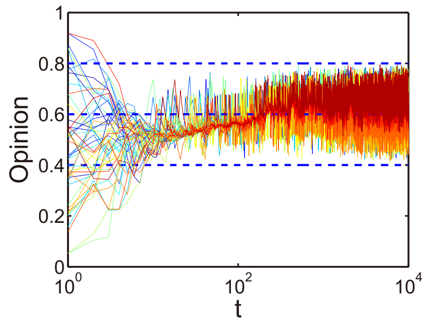

Fig. 5: Opinion evolution of the population with different stubborn individuals on ER network, where the dotted line in bold represents the opinion of stubborn agents

We also carry out the simulation on BA network[17], whose topology is shown in Fig.6. Assume the stubborn individuals are the three individuals with the largest degree, i.e. the three biggest nodes. The opinion evolution of the population is presented in Fig.7, from which it can be seen that some of the regular individuals converge to 0.4, 0.6, 0.8, respectively, while some of them fluctuate in the interval between 0.4 and 0.6. That is because in Fig.6 some of the regular individuals can reach only one stubborn individual, while a few others can really reach two stubborn individuals whose opinion happen to be 0.4 and 0.6. The results again agree with the conjecture in the previous section.

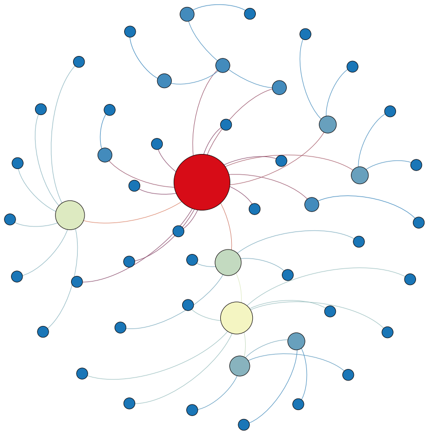

Fig. 6: The topology of BA scale-free network

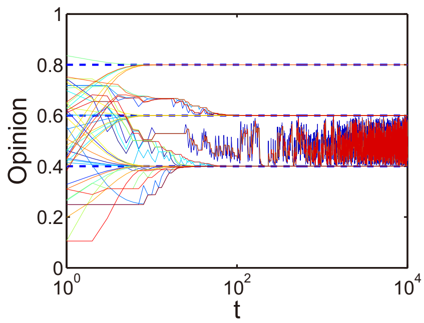

Fig. 7: Opinion evolution of the population on BA network, where the dotted line in bold represents the stubborn individuals' opinion

5 Conclusion

In this paper the dynamics of opinion evolution of the population with stubborn individuals over connected social network is investigated. Each individual picks up his partner based on his neighbor's popularity and closeness, and then updates his opinion by averaging his own opinion and his partner's in a weighted manner.

Theoretical analysis and numerical simulation show that both opinion consensus and fluctuation can be observed depending on whether the opinion of the stubborn individuals is different or not. When the stubborn individuals have the same initial opinion, the population can agree on the stubborn individuals' opinion eventually. Otherwise, the regular individuals' opinion may keep fluctuated in a interval, if they can visit at least two stubborn individuals with different opinion directly or indirectly through other regular individuals. The upper and lower bounds of the interval correspond to the largest and smallest opinion among the reachable stubborn individuals. The influence of the network topology on the dynamics of opinion formation is also analyzed. It is found that larger average degree and rewiring probability in SW network can accelerate the convergence speed. Similar effect is also observed in the random network as the connection probability becomes larger.

The result of this paper can shed some insight into uncovering the mechanism of opinion formation and the future work can look into the influence of external interference.

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