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Identification of Systems with Logarithmic Quantizer

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Abstract: The identification problem of systems with logarithmic quantized data is analyzed in this paper. A discrete-time system with an unknown parameter vector and Gaussian noise is considered. The output of the system is quantized by a logarithmic quantizer. The estimator of the unknown parameter vector is obtained using the quantile-based estimation. The convergence almost everywhere of the estimator and the upper bound of the estimation error are also analyzed, respectively.

Key Words: System identification, logarithmic quantizer, estimator, estimation error

1 Introduction

System identification of plants with quantized observations is significant in the area of instrumentation and measurement. In such problems, only limited information of the outputs is available, the systems have to be identified based on such quantized observations. The corresponding problem is called quantized identification [1]. Compared with traditional identification problem, the quantized identification problem is not able to access the original analog amplitude observations. Thus it can introduce new challenges in the areas of identification, control, system modeling, and so on.

In recent years, the problem of identifying signal parameters, after noise is added and quantization is performed, has constituted a vast body of literature. For example, in [2], a quantile-based estimator was presented, which was based on the Gauss–Markov theorem. [3] considered the problem of estimating the cumulative distribution function and probability density function of a random variable using data quantized by uniform and nonuniform quantizers. A simple estimator was proposed based on the empirical distribution function that also considered the values of the quantizer transition levels. [4] proposed a new estimator that used information about the values of the analog-to-digital converter (ADC) transition levels to improve the performance of conventional estimators. The estimator was based on the knowledge of the ADC transition levels, so that an initial calibration phase was necessary before actual parameter estimation. [5] was concerned with an estimation framework with scheduled measurements for a linear system. More results about this topic can be seen, for example, in [6]-[10].

In digital systems, quantization is usually realized a quantizer, which is categorized as static or dynamic. For static quantizer, it is a memoryless nonlinear function and assumes that data quantization is dependent on the data at each instant of time only [11-15]. In other hand, for dynamic quantizer, it uses memory and scales the quantization levels dynamically. Thus it can increase the region of attraction and attenuate the steady state limit cycle [16-20]. Each quantizer has its own advantages and limitations and it is hard to claim which is better. For static quantizer, one of the popular and frequently used is the logarithmic quantizer, which is firstly proposed in [11]. It was shown in [11] that for quadratic stabilization of a linear system using state feedback, the optimal static quantizer was a logarithmic quantizer and the coarsest quantization density was given explicitly in terms of the system's unstable poles. Though the study of quantized identification constitutes a vast body of the literature, the research of system identification using logarithmic quantizer data has not received enough attention. Most of the research about quantized identification is based on the uniform quantizer. Duo to the better efficiency of a logarithmic quantizer in terms of data rate for performance control than a uniform quantizer, the study of this problem is significant. The above motivates the research of the paper.

In this paper, we will mainly study the quantized identification problem of systems with logarithmic quantized data. The rest of this paper is organized as follows. In Section 2, we describe the system identification problem. Section 3 gives the main results of this paper, where the estimation of the interested parameter and the stochastic properties of the estimation are detailed analyzed. Some conclusions are also given in Section 4.

2 Problem Formulation

2.1 System Description

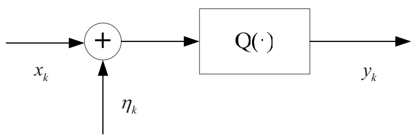

Figure 1. Configuration of system studied in this paper

The system considered in this paper is illustrated in Figure 1. For the system, x_k represents a discrete-time deterministic sequence with a vector parameter θ . $\eta_k\left(k=0,1,2,\cdots\right)$ is a zero-mean Gaussian noise sequence with variance σ^2 ($\sigma>0$). $Q(\cdot)$ is a quantizer. The output of

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the quantizer is denoted as y_k $(k=0,1,2,\cdots)$, i.e., y_k is expressed in the following

$$y_k = Q(x_k + \eta_k) \quad (1)$$

In this paper, suppose that x_k is a discrete-time linear system which is time-invariant and described as follows:

$$x_k = \sum_{i=0}^{\infty} a_i u_{k-i}, \ k=1,2,\cdots \quad (2)$$

where the system parameters a_i satisfy $\sum_{i=0}^{\infty}|a_i| < \infty$, $\{u_k\}$ is the input with $u_k = 0$ ($k<0$) and $|u_k| < u_{\max}$. Moreover, the system x_k is decomposed into two parts:

$$x_k = \sum_{i=0}^{\infty} a_i u_{k-i} = \sum_{i=0}^{m-1} a_i u_{k-i} + \sum_{i=m}^{\infty} a_i u_{k-i} \quad (3)$$

Define $\theta = [a_0, a_1, \cdots, a_{m-1}]^T$ is the modelled part, $\phi = [a_m, a_{m+1}, \cdots]^T$ is the unmodelled part, and the corresponding regressors are $\gamma_k = [u_k, u_{k-1}, \cdots, u_{k-m+1}]^T$ and $\tilde{\gamma}_k = [u_{k-m}, u_{k-m-1}, \cdots]^T$, respectively. Then the system x_k can be expressed as

$$x_k = \gamma_k^T \theta + \tilde{\gamma}_k^T \phi \quad (4)$$

2.2 Quantizer Description

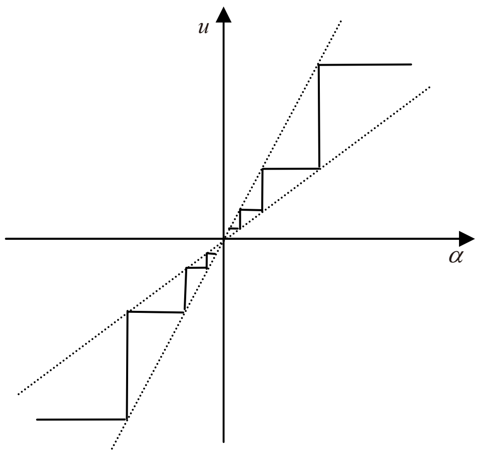

Figure 2. Logarithmic partition in the α, u plane [11]

For the system studied in Figure 1, suppose that $Q(\cdot)$ is a logarithmic quantizer with quantization density $\rho \in (0,1)$. It has the following quantization levels

$$U = \{\pm m_j : m_j = \rho^j \mu, j = 0, \pm 1, \pm 2, \cdots\} \cup \{0\} \quad (5)$$

where $\mu > 0$ is a scaling parameter. A small ρ means the coarse quantization and a large ρ implies the dense quantization. The quantizer $Q(\cdot)$ is depicted in Figure 2 and can be defined as follows:

$$Q(\varepsilon) = \begin{cases} \rho^j \mu, & if \ \dfrac{\rho^j \mu}{1+\delta} \le \varepsilon < \dfrac{\rho^j \mu}{1-\delta}, \ j = 0, \pm 1, \pm 2, \cdots \\ 0, & if \ \varepsilon = 0 \\ -Q(-\varepsilon), & if \ \varepsilon < 0 \end{cases} \quad (6)$$

where $\delta = \dfrac{1-\rho}{1+\rho}$. It also can be written in the following form

$$Q(\varepsilon) = \begin{cases} \rho^j \mu, & if \ \dfrac{\rho^j \mu(1+\rho)}{2} \le \varepsilon < \dfrac{\rho^{j-1} \mu(1+\rho)}{2} \\ & j = 0, \pm 1, \pm 2, \cdots \\ 0, & if \ \varepsilon = 0 \\ -Q(-\varepsilon), & if \ \varepsilon < 0 \end{cases} \quad (7)$$

An important feature of the logarithmic quantizer is that the quantizer can be bounded by a sector. The sector bound is described by the parameter δ. Specifically, for the logarithmic quantizer, the quantization error is

$$e_k = Q(x_k + \eta_k) - (x_k + \eta_k) = \Delta(x_k + \eta_k) \quad (8)$$

where $\Delta \in [-\delta, \delta]$. It is obvious that there are infinite thresholds for the logarithmic quantizer. Denoted the infinite thresholds as $C_1 < C_2 < \cdots < C_i < C_{i+1} < \cdots$, where C_i represents the *k*th quantization transition level in this paper. C_i has the form $C_i = \pm \dfrac{\rho^j \mu (1+\rho)}{2}$, where $j = 0, \pm 1, \pm 2, \cdots$.

According to the description of the logarithmic quantizer, it is obtained that

1) when $\varepsilon > 0$ and ε belongs to the interval $(C_i, C_{i+1}]$, the quantized value $Q(\varepsilon) = \dfrac{2C_i}{1+\rho}$.

2) when $\varepsilon < 0$ and ε belongs to the interval $[C_i, C_{i+1})$, the quantized value $Q(\varepsilon) = \dfrac{2\rho C_i}{1+\rho}$

Remark 1: From the description of $Q(\cdot)$, it is obvious that the logarithmic quantizer (7) is nonuniform and never overloaded.

Assume that N samples of the sequence $y_k = Q\left(\gamma_k^T \theta + \tilde{\gamma}_k^T \phi + \eta_k\right)$, which is defined in (1), are collected. The purpose of this paper is to estimate the unknown parameter θ from the N observations $y_1, y_2, \cdots, y_N$.

2.3 Quantile-Based Estimation Description

In this paper, the method of quantile-based estimation is used to identify the unknown parameter vector θ. Quantile-based estimation was proposed in reference [2]. It mainly deals with nonuniform quantized identification problem and has the advantage that it can remove the estimation bias. The main idea of quantile-based estimation will be exemplified according a simple system with single-bit quantizer in the following[2].

For the system depicted in Figure 1, assume that a constant disturbed by the zero-mean Gaussian noise is quantized and the quantizer is single-bit. The system can be expressed as follows

$$x_k = \theta, \ y_k = \begin{cases} 1, & \text{if } x_k + \eta_k \ge 0 \\ 0, & \text{if } x_k + \eta_k < 0 \end{cases}, \ k = 0, 1, 2, \cdots \quad (9)$$

Then the probability of $y_k = 1$ is

$$p = \Pr\{y_k = 1\} = \Pr\{\theta + \eta_k \ge 0\} = 1 - \Phi\left(\frac{-\theta}{\sigma}\right) \quad (10)$$

where $\Phi(\cdot)$ is the cumulative distribution function of a standard Gaussian random variable. By repeating the experiment several times, the probability p can be estimated by the percentage count $\hat{p}$ of the number of times that 1 is observed. When $\hat{p}$ is known, then

$$\hat{p} = 1 - \Phi\left(\frac{-\hat{\theta}}{\sigma}\right) \tag{11}$$

where $\hat{\theta}$ is the estimate of θ and can be expressed as follows

$$\hat{\theta} = -\sigma\Phi^{-1}(1-\hat{p}) \tag{12}$$

Then $\hat{\theta}$ is the quantile-based estimation of θ.

In this paper, the idea of quantile-based estimation is extended to the case of logarithmic quantizer. When x_k is defined in (2), $\theta = [a_0, a_1, \cdots, a_{m-1}]^T$ and $Q(\cdot)$ is the logarithmic quantizer, how to identify θ is the main problem we will concern in the following.

3 Main results

3.1 Estimation of θ

Denote $p_i = \Pr\{y_k \le s_i\}$, where s_i is the quantization value of the logarithmic quantizer (7). Thus the possible values of s_i are 0, $\pm\rho^j\mu$ ($j = 0, \pm 1, \pm 2, \cdots$). Then

$$p_i = \Pr\{y_k \le s_i\} = \Pr\left\{Q\left(\gamma_k^T\theta + \tilde{\gamma}_k^T\phi + \eta_k\right) \le s_i\right\} \tag{13}$$

According to the property of sector bound for the logarithmic quantizer,

$$Q\left(\gamma_k^T\theta + \tilde{\gamma}_k^T\phi + \eta_k\right) = (1+\Delta)\left(\gamma_k^T\theta + \tilde{\gamma}_k^T\phi + \eta_k\right),\ \Delta \in [-\delta, \delta] \tag{14}$$

Then

$$\begin{aligned} p_i &= \Pr\left\{(1+\Delta)\left(\gamma_k^T\theta + \tilde{\gamma}_k^T\phi + \eta_k\right) \le s_i\right\} \\ &= \Pr\left\{\gamma_k^T\theta + \tilde{\gamma}_k^T\phi + \eta_k \le \frac{s_i}{1+\Delta}\right\} \\ &= \Phi\left(\frac{1}{\sigma}\left(\frac{s_i}{1+\Delta} - \tilde{\gamma}_k^T\phi - \gamma_k^T\theta\right)\right) \end{aligned} \tag{15}$$

The estimator $\hat{p}^{(j)}$ of $p^{(j)}$ can be estimated according to the collecting N samples y_1, y_2, $\cdots$, y_N. It is obvious that $p^{(j)}$ is the average occurrence frequency of samples that is less than or equal to $s^{(j)}$. When N examples are collected, the estimator of $p^{(j)}$ can be obtained by $\hat{p}^{(j)} = \frac{1}{N}\sum_{i=1}^{N} I_{\{y_i \le s^{(j)}\}}$, where $I_{\{\}}$ is the indicator function. It can be described as follows:

$$I_{\{y_k \in A\}} = \begin{cases} 1, & \text{if } y_k \in A \\ 0, & \text{otherwise} \end{cases}$$

Letting

$$\hat{p}^{(j)} = \Phi\left(\frac{1}{\sigma}\left(\frac{s_i}{1+\Delta} - \tilde{\gamma}_k^T\phi - \gamma_k^T\theta\right)\right) \tag{16}$$

Since $\Phi^{-1}(\cdot)$ is invertible, then

$$\gamma_k^T\theta = \frac{s_i}{1+\Delta} - \tilde{\gamma}_k^T\phi - \sigma\Phi^{-1}\left(\hat{p}^{(j)}\right) \tag{17}$$

Two cases are categorized to analyze in the following.

Case 1: when the standard variance σ of the sequence η_k is known, by applying the LSE, an estimate of the parameter θ can be obtained as follows:

$$\hat{\theta} = \left(\gamma_k\gamma_k^T\right)^{-1}\gamma_k\left(\frac{s_i}{1+\Delta} - \tilde{\gamma}_k^T\phi - \sigma\Phi^{-1}\left(\hat{p}^{(j)}\right)\right) \tag{18}$$

Case 2: when σ is unknown, rewrite the expression (17) as follows:

$$\gamma_{kU}^T\theta_U = -\Phi^{-1}\left(\hat{p}^{(j)}\right) \tag{19}$$

where $\gamma_{kU} = \left[\gamma_k^T \quad \frac{s_i}{1+\Delta} - \tilde{\gamma}_k^T\phi\right]^T$ and $\theta_U = \left[\frac{\theta}{\sigma} \quad -\frac{1}{\sigma}\right]^T$. Using the LSE, an estimate of the parameter θ_U can be obtained as follows:

$$\hat{\theta}_U = \left[\frac{\hat{\theta}}{\hat{\sigma}} \quad -\frac{1}{\hat{\sigma}}\right]^T = -\left(\gamma_{kU}\gamma_{kU}^T\right)^{-1}\gamma_{kU}\Phi^{-1}\left(\hat{p}^{(j)}\right) \tag{20}$$

By doing some simple transformation of $\hat{\theta}_U$, the estimator $\hat{\theta}$ is obtained.

The estimators (19) and (20) of the unknown parameter θ contain Δ, which belongs to the interval $[-\delta, \delta]$. How to choose Δ is what we will study in the following. In fact, according to the sector bound property of the logarithmic quantizer, when the quantization value $s_i > 0$, if $y_k \le s_i$, then $\gamma_k^T\theta + \tilde{\gamma}_k^T\phi + \eta_k < \frac{\rho^j\mu}{1-\delta}$, i.e., $\Delta = \delta$. Otherwise, $\Delta = -\delta$.

Thus

If $s_i > 0$, $p_i = \Phi\left(\frac{1}{\sigma}\left(\frac{(1+\rho)s_i}{2} - \tilde{\gamma}_k^T\phi - \gamma_k^T\theta\right)\right)$;

If $s_i < 0$, $p_i = \Phi\left(\frac{1}{\sigma}\left(\frac{(1+\rho)s_i}{2\rho} - \tilde{\gamma}_k^T\phi - \gamma_k^T\theta\right)\right)$.

The algorithm of the estimator for the unknown parameter θ can be summarized as follows:

Algorithm 1.

Case 1: σ is known.

If $s_i > 0$, $$\hat{\theta} = \left(\gamma_k\gamma_k^T\right)^{-1}\gamma_k\left(\frac{(1+\rho)s_i}{2} - \tilde{\gamma}_k^T\phi - \sigma\Phi^{-1}\left(\hat{p}^{(j)}\right)\right) \tag{21}$$

If $s_i < 0$, $$\hat{\theta} = \left(\gamma_k\gamma_k^T\right)^{-1}\gamma_k\left(\frac{(1+\rho)s_i}{2\rho} - \tilde{\gamma}_k^T\phi - \sigma\Phi^{-1}\left(\hat{p}^{(j)}\right)\right) \tag{22}$$

Case 2: σ is unknown.

If $s_i > 0$, $$\begin{aligned} \hat{\theta}_U &= \left[\frac{\hat{\theta}}{\hat{\sigma}} \quad -\frac{1}{\hat{\sigma}}\right]^T \\ &= -\left(\begin{bmatrix} \gamma_k \\ \frac{(1+\rho)s_i}{2} - \tilde{\gamma}_k^T\phi \end{bmatrix}\begin{bmatrix} \gamma_k^T & \frac{(1+\rho)s_i}{2} - \tilde{\gamma}_k^T\phi \end{bmatrix}\right)^{-1} \\ &\quad \cdot \begin{bmatrix} \gamma_k \\ \frac{(1+\rho)s_i}{2} - \tilde{\gamma}_k^T\phi \end{bmatrix}\Phi^{-1}\left(\hat{p}^{(j)}\right) \end{aligned} \tag{23}$$

If $s_i < 0$, $$\begin{aligned} \hat{\theta}_U &= \left[\frac{\hat{\theta}}{\hat{\sigma}} \quad -\frac{1}{\hat{\sigma}}\right]^T \\ &= -\left(\begin{bmatrix} \gamma_k \\ \frac{(1+\rho)s_i}{2} - \tilde{\gamma}_k^T\phi \end{bmatrix}\begin{bmatrix} \gamma_k^T & \frac{(1+\rho)s_i}{2\rho} - \tilde{\gamma}_k^T\phi \end{bmatrix}\right)^{-1} \end{aligned}$$

$$\cdot \begin{bmatrix} \gamma_k \\ \frac{(1+\rho)s_i}{2\rho} - \tilde{\gamma}_k^T \phi \end{bmatrix} \Phi^{-1}\left(\hat{p}^{(j)}\right) \tag{24}$$

3.2 Stochastic property of the estimator

Some definitions and lemmas are presented before analyzing the stochastic property of the estimator.

Definition 1(Convergence almost everywhere)[21]. A sequence $\{\xi_n(\omega)\}$ is said to converge almost everywhere to $\xi(\omega)$ if $\Pr\left\{\lim_{n\to\infty}\xi_n(\omega)=\xi(\omega)\right\}=1$. Denote it as $\xi_n(\omega)\xrightarrow{a.s.}\xi(\omega)$.

Lemma 1(Kolmogorov strong law of large numbers)[21]. If the independent random variables ξ_1, ξ_2, $\cdots$ are identically distributed with a common law $L(\xi)$, then $\frac{1}{n}\sum_{i=1}^{n}\xi_i \xrightarrow{a.s.} c$, where c is finite, if and only if $\mathrm{E}(\xi)<\infty$; and then c=$\mathrm{E}(\xi)$.

Lemma 2(Chernoff's bound)[22]. For any $t>0$ and the random variables X_1, X_2, $\cdots$, X_n, then

$$\Pr\left\{\sum_{i=1}^{n}X_i \ge a\right\} \le \min_{t>0}\exp(-ta)\prod_{i=1}^{n}\mathrm{E}\left[\exp(tX_i)\right] \tag{25}$$

and

$$\Pr\left\{\sum_{i=1}^{n}X_i \le a\right\} \le \min_{t>0}\exp(ta)\prod_{i=1}^{n}\mathrm{E}\left[\exp(-tX_i)\right] \tag{26}$$

In the following, we will take the case that σ is known and $s_i>0$ as an example to analyze the convergence of the estimator $\hat{\theta}_N$ and the upper bound of the estimation error. For other cases that σ is unknown or $s_i>0$, the analysis method is similar and the results are omitted in this paper.

Theorem 1. Suppose that σ is known and $\gamma_k\gamma_k^T$ is nonsingular. When $s_i>0$, then the estimator $\hat{\theta}$ obtained in (21) satisfies that $\hat{\theta}\xrightarrow{a.s.}\theta$.

Proof. It is obvious that $\left\{I_{\left\{y_i\le s^{(j)}\right\}}\right\}$ ($i=1,2,\cdots N$) is a sequence of i.i.d. Bernoulli random variables. Then

$$\mathrm{E}\left\{I_{\left\{y_i\le s^{(j)}\right\}}\right\}=1\times\Pr\left\{y_i\le s^{(j)}\right\}+0\times\Pr\left\{y_i>s^{(j)}\right\}p^{(j)}$$

$$\mathrm{E}\left(\hat{p}^{(j)}\right)=\frac{1}{N}\sum_{i=1}^{N}\mathrm{E}\left\{I_{\left\{y_i\le s^{(j)}\right\}}\right\}=p^{(j)}$$

Then according to the Kolmogorov strong law of large numbers,

$$\hat{p}^{(j)}=\frac{1}{N}\sum_{i=1}^{N}I_{\left\{y_i\le s^{(j)}\right\}}\xrightarrow{a.s.}p^{(j)}$$

Since σ is known and $\gamma_k\gamma_k^T$ is nonsingular, if $s_i>0$, then (21) holds. The continuity of $\Phi^{-1}(\cdot)$ implies that $\Phi^{-1}\left(\hat{p}^{(j)}\right)\xrightarrow{a.s.}\Phi^{-1}\left(p^{(j)}\right)$, i.e.,

$$\Phi^{-1}\left(\hat{p}^{(j)}\right)\xrightarrow{a.s.}\frac{1}{\sigma}\left(\frac{(1+\rho)s_i}{2}-\tilde{\gamma}_k^T\phi-\gamma_k^T\theta\right)$$

Then $\frac{(1+\rho)s_i}{2}-\tilde{\gamma}_k^T\phi-\sigma\Phi^{-1}\left(\hat{p}^{(j)}\right)\xrightarrow{a.s.}\gamma_k^T\theta$, i.e.,

$$\hat{\theta}=\left(\gamma_k\gamma_k^T\right)^{-1}\gamma_k\left(\frac{(1+\rho)s_i}{2}-\tilde{\gamma}_k^T\phi-\sigma\Phi^{-1}\left(\hat{p}^{(j)}\right)\right)\xrightarrow{a.s.}\theta$$

The proof is completed.

Theorem 2. Suppose that σ is known and $\gamma_k\gamma_k^T$ is nonsingular. When $s_i>0$, then for any $\varepsilon>0$,

$$\begin{aligned}&\Pr\left\{\left\|\hat{\theta}-\theta\right\|\ge\varepsilon\right\}\\ &\le g_1\left\{N\Phi\left[\frac{1}{\sigma}\left(\frac{(1+\rho)s_i}{2}-\tilde{\gamma}_k^T\phi-\gamma_k^T\theta-\frac{\varepsilon}{\left\|\left(\gamma_k\gamma_k^T\right)^{-1}\gamma_k\right\|}\right)\right]\right\}\\ &+g_2\left\{N\Phi\left[\frac{1}{\sigma}\left(\frac{(1+\rho)s_i}{2}-\tilde{\gamma}_k^T\phi-\gamma_k^T\theta+\frac{\varepsilon}{\left\|\left(\gamma_k\gamma_k^T\right)^{-1}\gamma_k\right\|}\right)\right]\right\}\end{aligned} \tag{27}$$

where

$$g_1(x)=\min_{t>0}\exp(-tx)\prod_{k=0}^{N-1}\mathrm{E}\left[\exp\left(t\hat{y}_k\right)\right]$$

$$g_2(x)=\min_{t>0}\exp(tx)\prod_{k=0}^{N-1}\mathrm{E}\left[\exp\left(-t\hat{y}_k\right)\right]$$

Proof. Since σ is known and $\gamma_k\gamma_k^T$ is nonsingular, if $s_i>0$, then (21) holds. Thus

$$\begin{aligned}\hat{\theta}-\theta&=\left(\gamma_k\gamma_k^T\right)^{-1}\gamma_k\left(\frac{(1+\rho)s_i}{2}-\tilde{\gamma}_k^T\phi-\sigma\Phi^{-1}\left(\hat{p}^{(j)}\right)\right)-\theta\\ &=\left(\gamma_k\gamma_k^T\right)^{-1}\gamma_k\left(\frac{(1+\rho)s_i}{2}-\tilde{\gamma}_k^T\phi-\sigma\Phi^{-1}\left(\hat{p}^{(j)}\right)-\gamma_k^T\theta\right)\end{aligned}$$

Therefore

$$\left\|\hat{\theta}-\theta\right\|\le\left\|\left(\gamma_k\gamma_k^T\right)^{-1}\gamma_k\right\|\left|\frac{(1+\rho)s_i}{2}-\tilde{\gamma}_k^T\phi-\sigma\Phi^{-1}\left(\hat{p}^{(j)}\right)-\gamma_k^T\theta\right|$$

For any $\varepsilon>0$, it is obtained that

$$\begin{aligned}&\Pr\left\{\left\|\hat{\theta}-\theta\right\|\ge\varepsilon\right\}\\ &\le\Pr\left\{\left\|\left(\gamma_k\gamma_k^T\right)^{-1}\gamma_k\right\|\left|\frac{(1+\rho)s_i}{2}-\tilde{\gamma}_k^T\phi-\sigma\Phi^{-1}\left(\hat{p}^{(j)}\right)-\gamma_k^T\theta\right|\ge\varepsilon\right\}\\ &=\Pr\left\{\frac{(1+\rho)s_i}{2}-\tilde{\gamma}_k^T\phi-\sigma\Phi^{-1}\left(\hat{p}^{(j)}\right)-\gamma_k^T\theta\ge\frac{\varepsilon}{\left\|\left(\gamma_k\gamma_k^T\right)^{-1}\gamma_k\right\|}\right\}\\ &+\Pr\left\{\frac{(1+\rho)s_i}{2}-\tilde{\gamma}_k^T\phi-\sigma\Phi^{-1}\left(\hat{p}^{(j)}\right)-\gamma_k^T\theta\le-\frac{\varepsilon}{\left\|\left(\gamma_k\gamma_k^T\right)^{-1}\gamma_k\right\|}\right\}\\ &=\Pr\left\{\Phi^{-1}\left(\hat{p}^{(j)}\right)\le\frac{1}{\sigma}\left(\frac{(1+\rho)s_i}{2}-\tilde{\gamma}_k^T\phi-\gamma_k^T\theta-\frac{\varepsilon}{\left\|\left(\gamma_k\gamma_k^T\right)^{-1}\gamma_k\right\|}\right)\right\}\\ &+\Pr\left\{\Phi^{-1}\left(\hat{p}^{(j)}\right)\ge\frac{1}{\sigma}\left(\frac{(1+\rho)s_i}{2}-\tilde{\gamma}_k^T\phi-\gamma_k^T\theta+\frac{\varepsilon}{\left\|\left(\gamma_k\gamma_k^T\right)^{-1}\gamma_k\right\|}\right)\right\}\end{aligned}$$

According to the monotonicity and continuity of $\Phi(\cdot)$, we obtain that

$$\Pr\left\{\left\|\hat{\theta}-\theta\right\|\geq\varepsilon\right\}$$

$$=\Pr\left\{\hat{p}^{(j)}\leq\Phi\left[\frac{1}{\sigma}\left(\frac{(1+\rho)s_i}{2}-\tilde{\gamma}_k^T\phi-\gamma_k^T\theta-\frac{\varepsilon}{\left\|\left(\gamma_k\gamma_k^T\right)^{-1}\gamma_k\right\|}\right)\right]\right\}$$

$$+\Pr\left\{\hat{p}^{(j)}\geq\Phi\left[\frac{1}{\sigma}\left(\frac{(1+\rho)s_i}{2}-\tilde{\gamma}_k^T\phi-\gamma_k^T\theta+\frac{\varepsilon}{\left\|\left(\gamma_k\gamma_k^T\right)^{-1}\gamma_k\right\|}\right)\right]\right\}$$

$$=\Pr\left\{\sum_{i=1}^{N}I_{\left\{y_i\leq s^{(j)}\right\}}\leq N\Phi\left[\frac{1}{\sigma}\left(\frac{(1+\rho)s_i}{2}-\tilde{\gamma}_k^T\phi-\gamma_k^T\theta-\frac{\varepsilon}{\left\|\left(\gamma_k\gamma_k^T\right)^{-1}\gamma_k\right\|}\right)\right]\right\}$$

$$+\Pr\left\{\sum_{i=1}^{N}I_{\left\{y_i\leq s^{(j)}\right\}}\geq N\Phi\left[\frac{1}{\sigma}\left(\frac{(1+\rho)s_i}{2}-\tilde{\gamma}_k^T\phi-\gamma_k^T\theta+\frac{\varepsilon}{\left\|\left(\gamma_k\gamma_k^T\right)^{-1}\gamma_k\right\|}\right)\right]\right\}$$

From the Chernoff's bound, it follows that

$$\Pr\left\{\sum_{i=1}^{N}I_{\left\{y_i\leq s^{(j)}\right\}}\leq N\Phi\left[\frac{1}{\sigma}\left(\frac{(1+\rho)s_i}{2}-\tilde{\gamma}_k^T\phi-\gamma_k^T\theta-\frac{\varepsilon}{\left\|\left(\gamma_k\gamma_k^T\right)^{-1}\gamma_k\right\|}\right)\right]\right\}$$

$$\leq g_1\left\{N\Phi\left[\frac{1}{\sigma}\left(\frac{(1+\rho)s_i}{2}-\tilde{\gamma}_k^T\phi-\gamma_k^T\theta-\frac{\varepsilon}{\left\|\left(\gamma_k\gamma_k^T\right)^{-1}\gamma_k\right\|}\right)\right]\right\}$$

$$\Pr\left\{\sum_{i=1}^{N}I_{\left\{y_i\leq s^{(j)}\right\}}\geq N\Phi\left[\frac{1}{\sigma}\left(\frac{(1+\rho)s_i}{2}-\tilde{\gamma}_k^T\phi-\gamma_k^T\theta+\frac{\varepsilon}{\left\|\left(\gamma_k\gamma_k^T\right)^{-1}\gamma_k\right\|}\right)\right]\right\}$$

$$\leq g_2\left\{N\Phi\left[\frac{1}{\sigma}\left(\frac{(1+\rho)s_i}{2}-\tilde{\gamma}_k^T\phi-\gamma_k^T\theta+\frac{\varepsilon}{\left\|\left(\gamma_k\gamma_k^T\right)^{-1}\gamma_k\right\|}\right)\right]\right\}$$

Thus the expression (27) holds. This completes the proof.

4 Conclusion

In this paper, we have studied the problem of system identification using logarithmic quantized data. We have considered the discrete-time sequence with an unknown vector parameter. Using the sector bound property of the logarithmic quantizer and the method of quantile-based estimation, the estimator of the unknown vector parameter has been obtained. Then the algorithm of how to calculate the estimator has been also given. Furthermore, the convergence of the estimator has been analyzed and the upper bound of the estimation error has been obtained.

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